Combinatória I: Lista 2

1. Use the Janson inequalities and Harris' lemma to prove that if $pn = c$, then

$$\mathbb{P}\big(K_3 \subset G(n,p)\big) \to 1 - e^{-c^3/6}$$

as $n \to \infty$.

2. Show that, for every graph H, there exists $\delta > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. If G is a graph on n vertices with

$$e(G) > (1-\delta)\binom{n}{2},$$

then in every r-colouring of $E(G)$ there are at least $\delta n^{v(H)}$ monochromatic copies of H.

3. Prove that there exist triangle-free graphs with arbitrarily large chromatic number.

4. Prove that there exists a tournament of order n containing at least $2^{-n}(n-1)!$ directed Hamiltonian cycles. (An ordering $(v_1, \dots, v_n)$ of the vertices is a directed Hamiltonian cycle if v_i beat v_{i+1} for every $i \in \mathbb{Z}_n$.)

5. Say that a k-uniform hypergraph G is said to be 2-*colourable* if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges in a k-uniform hypergraph that is not 2-colourable.
(a) By considering a random colouring, show that $b(k) \geqslant 2^{k-1}$.
(b) By considering a random hypergraph, prove an upper bound for $b(k)$.

6. Show that any finite set A of integers contains a sum-free subset of size at least $|A|/3$.[1]

7. Let $(e_1, \dots, e_m)$ be an arbitrary ordering of the edges of a graph G on n vertices. Show that there exists an increasing walk (in this ordering) of length at least $d = 2m/n$.

8. Show that the number of triangle-free graphs on n vertices is $2^{n^2/4 + o(n^2)}$.

9. Show that almost all triangle-free graphs are $o(n^2)$-close to being bipartite.

10. Prove Thomassen's theorem: for every $\alpha > 0$ there exists C such that every triangle-free graph with minimum degree $\delta(G) > (1/3 + \alpha)n$ has chromatic number at most C.[2]

11. Show that $\mathbf{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ for every sufficiently large n.

[1] Hint: multiply by a random number modulo some prime p, and consider the middle third.
[2] Hint: show that for each $I \subset [k]$, the set $X_I := \{v : |N(v) \cap V_i| \geqslant \beta |V_i| \Leftrightarrow i \in I\}$ is independent.