Combinatória I 2019: Lista 2

1. Determine the threshold for the event $\{K_r \subset G(n,p)\}$. Is the threshold sharp?

2. Prove that if T is a tree with k vertices, then $r(K_r, T) = (r-1)(k-1)+1$.

3. Prove that $R(3,k) \geqslant k^{1+c}$ for some $c > 0$.

4. Say that a k-uniform hypergraph G is said to be 2-*colourable* if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges in a k-uniform hypergraph that is not 2-colourable.
(a) By considering a random colouring, show that $b(k) \geqslant 2^{k-1}$.
(b) By considering a random hypergraph, prove an upper bound for $b(k)$.

5. Show that, for every graph H, there exists $\delta > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. If G is a graph on n vertices with

$$e(G) > (1-\delta)\binom{n}{2},$$

then in every r-colouring of $E(G)$ there are at least $\delta n^{v(H)}$ monochromatic copies of H.

6. Suppose we are given n points and n lines in the plane. Show that there are at most $n^{3/2}$ point-line incidences.

7. Prove that there exists a tournament of order n containing at least $2^{-n}(n-1)!$ directed Hamiltonian cycles. (An ordering $(v_1, \ldots, v_n)$ of the vertices is a directed Hamiltonian cycle if v_i beat v_{i+1} for every $i \in \mathbb{Z}_n$.)

8. Let $(e_1, \ldots, e_m)$ be an arbitrary ordering of the edges of a graph G on n vertices. Show that there exists an increasing walk (in this ordering) of length at least $d = 2m/n$.

9. Prove that if $p \gg \frac{\log n}{n}$ then with high probability $G(n,p)$ contains a perfect matching.

10. Show that the number of triangle-free graphs on n vertices is $2^{n^2/4+o(n^2)}$.

11. Show that almost all triangle-free graphs are $o(n^2)$-close to being bipartite.

12. Use the triangle removal lemma to prove that for every 3-uniform hypergraph $\mathcal{H}$ with at least αn^2 edges (where $\alpha > 0$ is fixed and n is sufficiently large), there exists a set of 6 vertices that contains at least 3 edges of $\mathcal{H}$.

13. Show that $\mathbf{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ for every sufficiently large n.