Combinatória I 2017: Lista 2

1. Show that $R(3, k) \geqslant k^{4/3}$ for all sufficiently large k.

2. Prove that there exists a tournament of order n containing at least $2^{-n}(n-1)!$ directed hamiltonian cycles. (An ordering $(v_1, \ldots, v_n)$ of the vertices is a directed hamiltonian cycle if v_i beat v_{i+1} for every $i \in \mathbb{Z}_n$.)

3. Say that a k-uniform hypergraph G is said to be 2-*colourable* if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges in a k-uniform hypergraph that is not 2-colourable.

(a) Show that $b(2) = 3$.

(b) By considering a random colouring, show that $b(k) \geqslant 2^{k-1}$.

(c) By considering a random hypergraph, prove an upper bound for $b(k)$.

4. Use Szemerédi's regularity lemma to show that, for every $\delta > 0$, the number of triangle-free graphs on n vertices is at most $2^{n^2/4+\delta n^2}$ for all sufficiently large n.

5. Use Szemerédi's regularity lemma to show that for every $\alpha > 0$, there exists $\delta = \delta(\alpha) > 0$ such that the following holds: the number of triangle-free graphs that are αn^2-far from being bipartite is at most $2^{n^2/4-\delta n^2}$ for all sufficiently large n.

6. Use the triangle removal lemma to prove that for every 3-uniform hypergraph $\mathcal{H}$ with at least αn^2 edges (where $\alpha > 0$ is fixed and n is sufficiently large), there exists a set of 6 vertices that contains at least 3 edges of $\mathcal{H}$.

7. Prove the dependent random choice lemma, and deduce that

$$r(Q_k) \leqslant 2^{3k}.$$