Combinatória I: Lista 1

1. Show that every graph (of order at least 2) has two vertices of the same degree. Show also that there do *not* necessarily exist three vertices of the same degree.

2. By defining, and calculating the expectation of, a suitable random variable, show that every graph G has a bipartite subgraph with at least $e(G)/2$ edges.

3. Show that every planar graph has chromatic number at most 6, and every triangle-free planar graph has chromatic number at most 4.

4. Show that if T is a tree with k vertices and G is a graph with minimum degree $k-1$, then $T \subset G$. Deduce that $r(K_3, T) = 2k - 1$.

5. Show that every graph of average degree d contains a subgraph of minimum degree at least $d/2$. Deduce that $\mathbf{ex}(n, T) \leqslant (k-1)n$ for every tree T with k vertices.

6. Let $T_1, \ldots, T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex in all the T_i.

7. Let $R_r(3)$ denote the r-colour Ramsey number of a triangle. Show that

$$2^r \leqslant R_r(3) \leqslant 3 \cdot r!$$

Show moreover that $R_r(3) \geqslant 5^{r/2}$.

8. Recall that $\alpha(G)$ denotes the size of the largest independent set in G. Show that, for every graph G,

$$\alpha(G) \geqslant \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

9. Let $C(s)$ be the smallest n such that every connected graph on n vertices has, as an *induced* subgraph, either a complete graph K_s, a star $K_{1,s}$ or a path P_s of length s.

Show that $C(s) \leqslant R(s)^s$, where $R(s)$ is the Ramsey number of s.

10. Let G be a (not necessarily planar) graph with $|G| = n$ and $e(G) = m$. Suppose that G is drawn in the plane, but with edges allowed to cross. Let t be the number of pairs of edges which cross. Show that $t \geqslant m - 3n + 6$.

Suppose now $m \geqslant 4n$. By considering a random set $W \subset V(G)$ containing each vertex of G independently with probability $4n/m$, show that in fact $t \geqslant m^3/64n^2$.

11. Show that there is an infinite set S of positive integers such that the sum of any two distinct elements of S has an even number of distinct prime factors.